FPT algorithm for a generalized cut problem and some applications

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Cut problem given a graph, find a minimum (vertex or edge) cutset whose removal makes the graph satisfy some separation property.

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Cut problem given a graph, find a minimum (vertex or edge) cutset whose removal makes the graph satisfy some separation property.

- MIN CUT: polynomial by classical max-flow min-cut theorem.
- MULTIWAY CUT: FPT by using important separators. [Marx '06]
- MULTICUT: Finally, FPT. [Marx, Razgon + Bousquet, Daligault, Thomassé '10]
- STEINER CUT: Improved FPT algorithm by using randomized contractions. [Chitnis, Cygan, Hajiaghayi, Pilipczuk^2 '12]
- MIN BISECTION: Finally, FPT. [Cygan, Lokshtanov, Pilipczuk^2, Saurabh '13]

We introduce a new cut problem

- A new cut problem: List Allocation (to be defined in two slides).

Theorem

The List Allocation *problem is* FPT.

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The List Allocation *problem is* FPT*.*

- List Allocation generalizes, in particular, Multiway Cut.
- General enough so that several other problems can be reduced to it:
  - ⋆ FPT algorithm for a parameterization of Digraph Homomorphism.
  - ⋆ FPT algorithm for the Min-Max Graph Partitioning problem.
  - ⋆ FPT 2-approximation for Tree-cut width.

Before defining the problem: allocations

- An *r*-allocation of a set S is an r-tuple $\mathcal{V} = (V_1, \ldots, V_r)$ of possibly empty pairwise disjoint subsets of S whose union is S.
- Elements of $\mathcal{V}$: parts of $\mathcal{V}$.
- We denote by $\mathcal{V}^{(i)}$ the i-th part of $\mathcal{V}$, i.e., $\mathcal{V}^{(i)} = V_i$.

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- We denote by $\mathcal{V}^{(i)}$ the i-th part of $\mathcal{V}$, i.e., $\mathcal{V}^{(i)} = V_i$.
- Let $G = (V, E)$ be a graph and let $\mathcal{V}$ be an r-allocation of V:

  $|\delta(\mathcal{V}^{(i)}, \mathcal{V}^{(j)})|$: #edges in G with one endpoint in $\mathcal{V}^{(i)}$ and one in $\mathcal{V}^{(j)}$.

Definition of the problem: List Allocation

List Allocation

Input: A tuple $I = (G, r, \lambda, \alpha)$, where G is an n-vertex graph, $r \in \mathbb{Z}_{\geqslant 1}$, $\lambda : V(G) \to 2^{[r]}$, and $\alpha : \binom{[r]}{2} \to \mathbb{Z}_{\geqslant 0}$.

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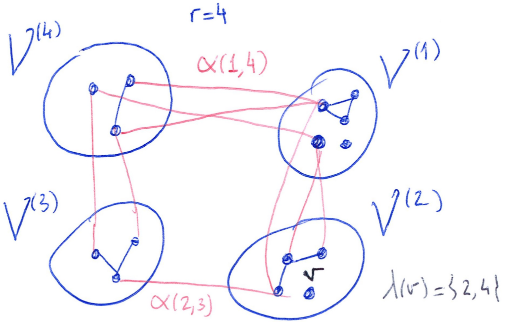
Parameter: $k = \sum \alpha$.

Question: Decide whether there exists an r-allocation $\mathcal{V}$ of $V(G)$ s.t.

- $\forall \{i, j\} \in \binom{[r]}{2}$, $|\delta(\mathcal{V}^{(i)}, \mathcal{V}^{(j)})| = \alpha(i, j)$ and
- $\forall v \in V(G)$, if $v \in \mathcal{V}^{(i)}$ then $i \in \lambda(v)$.



High-level ideas of the FPT algorithm

- Strongly inspired by the technique of randomized edge contraction.

[Chitnis, Cygan, Hajiaghayi, Pilipczuk[2] '12]

- We use a series of FPT reductions:

Problem A $\xrightarrow{\text{FPT}}$ Problem B: If problem B is FPT, then problem A is FPT.

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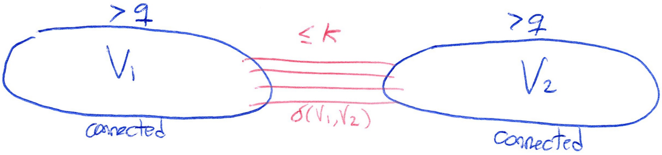
- At some steps, we obtain instances whose size is bounded by some function $f(k)$.
- Then we will use that the List Allocation problem is in XP:

Lemma

There exists an algorithm that, given an instance $I = (G, r, \lambda, \alpha)$ of List Allocation, *computes all possible solutions in time $n^{O(k)} \cdot r^{O(k+\ell)}$, where ℓ is the number of connected components of G.*

Some preliminaries

- Let G be a connected graph. A partition (V_1, V_2) of $V(G)$ is a (q, k)-separation if $|V_1|, |V_2| > q$, $|\delta(V_1, V_2)| \leqslant k$, and $G[V_1]$ and $G[V_2]$ are both connected.



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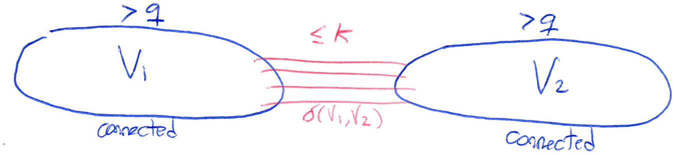

- A graph G is (q, k)-connected if it does not contain any $(q, k-1)$-separation.

Lemma (Chitnis, Cygan, Hajiaghayi, Pilipczuk[2] '12)

There exists an algorithm that given a n-vertex connected graph G and two integers q, k, either finds a (q, k)-separation, or reports that no such separation exists, in time $(q+k)^{O(\min\{q,k\})} n^3 \log n$.

Series of FPT reductions

List Allocation (LA)

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$\downarrow$ FPT

Connected List Allocation (CLA)

Same input + graph G is connected and $r \leqslant 2k$

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Same input + graph G is $(f_1(k), k+1)$-connected, for $f_1(k) := 2^k \cdot (2k)^{2k}$

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Claim (Unique big part)

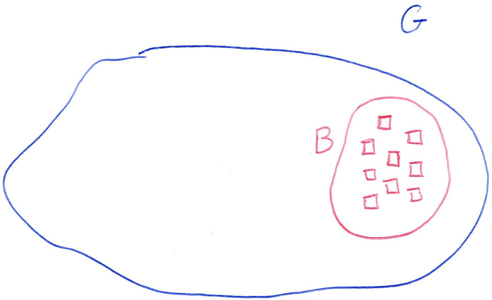
For any solution $\mathcal{V}$ of HCLA there exists a unique index $j \in [r]$ such that

$$\sum_{i \in [r] \setminus j} |\mathcal{V}^{(i)}| \leqslant k \cdot f_1(k).$$

- Part $\mathcal{V}^{(j)}$ is called the big part.

Reduction from CLA to HCLA: we shrink the graph

- We apply to G the following recursive algorithm `shrink`, which receives a graph G and a boundary set B with $|B| \leqslant 2k$ (start with $B = \emptyset$):



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     - W.l.o.g. let V_1 be the part with the smallest number of boundary vertices, and let B' be the new boundary: so $|B'| \leqslant 2k$.
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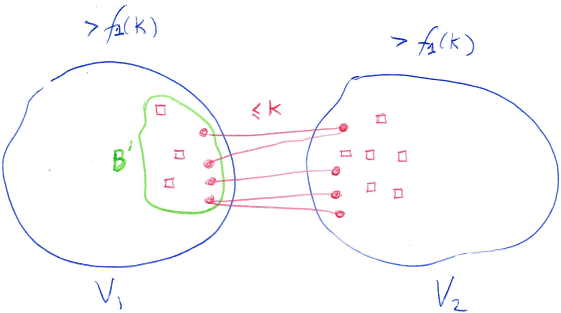

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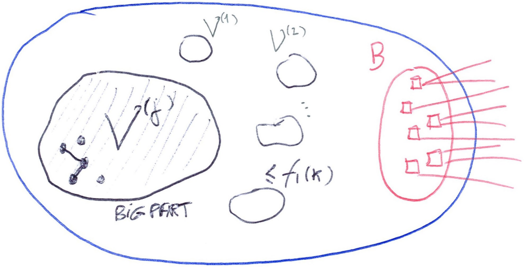
  2. Otherwise, find a set of "indistinguishable"' vertices, and identify them.

  Idea We generate all partial solutions in the boundary, and for each of them we compute a solution of HCLA, using our "black box".



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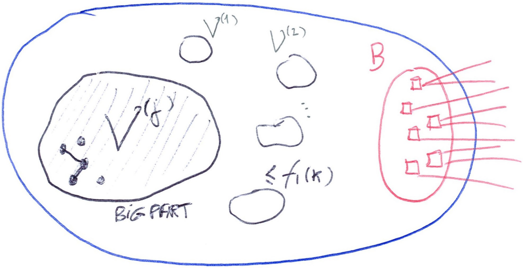
  2. Otherwise, find a set of "indistinguishable"' vertices, and identify them.

  Idea By the high connectivity (Claim), each such solution has a unique big part $\mathcal{V}^{(j)}$: indistinguishable vertices for this behavior.



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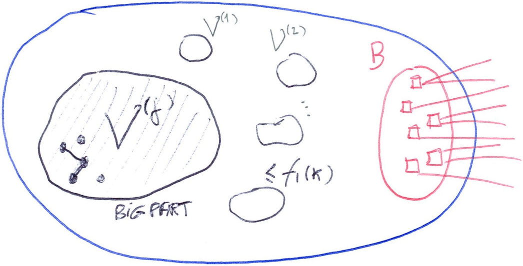
   Idea If the graph is big enough, there are vertices that are indistinguishable for all behaviors $\Rightarrow$ identify them. Return the graph.



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Lemma

The above algorithm returns in FPT time an equivalent instance of CLA of size at most $f_2(k) := k \cdot (f_1(k))^2 + 2k + 2$. (Then we apply the XP algorithm.)

Series of FPT reductions

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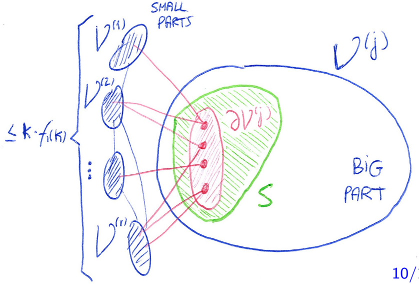
Split Highly Connected List Allocation (SHCLA)

Same input + set $S \subseteq V(G)$ and a solution $\mathcal{V}$ additionally needs to satisfy that if $j \in [r]$ is such that $\mathcal{V}^{(j)}$ is the big part of $\mathcal{V}$, then

$$\partial\mathcal{V}^{(j)} \subseteq S \subseteq \mathcal{V}^{(j)}.$$

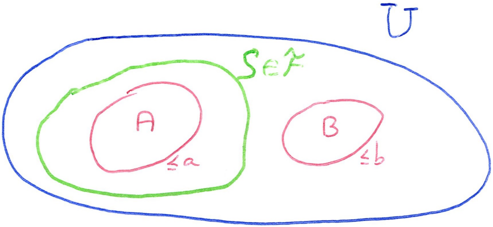


Crucial ingredient: Splitter Lemma

- Splitters were first introduced by [Naor, Schulman, Srinivasan '95]
- We use the following deterministic version:

Lemma (Chitnis, Cygan, Hajiaghayi, Pilipczuk^2 '12)

There exists an algorithm that given a set U of size n and two integers $a, b \in [0, n]$, outputs a set $\mathcal{F} \subseteq 2^U$ where $|\mathcal{F}| = (a+b)^{O(\min\{a,b\})} \cdot \log n$ such that for every two sets $A, B \subseteq U$, where $A \cap B = \emptyset$, $|A| \leqslant a$, $|B| \leqslant b$, there exists a set $S \in \mathcal{F}$ where $A \subseteq S$ and $B \cap S = \emptyset$, in $(a+b)^{O(\min\{a,b\})} \cdot n \log n$ steps.



Reduction from HCLA to SHCLA: we use splitters

- We use the Splitter Lemma with universe $U = V(G)$, $a = k$, and $b = k \cdot f_1(k)$, obtaining a family $\mathcal{F}$ of subsets of $V(G)$.

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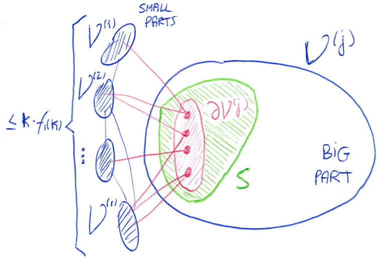
- Idea We want a set $S \subseteq V(G)$ that "splits" these two sets:

$$A = \partial \mathcal{V}^{(j)} \quad \text{and} \quad B = \bigcup_{i \in [r] \setminus \{j\}} \mathcal{V}^{(i)}.$$

For some $j \in [r]$: $|A| \leqslant k$ and $|B| \leqslant k \cdot f_1(k)$ (by the Claim).



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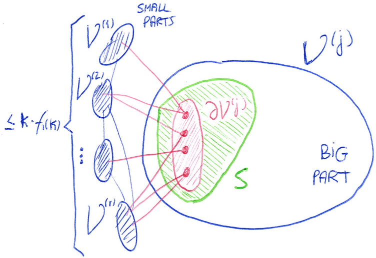

- It holds that I is a YES-instance of HCLA if and only if for some $S \in \mathcal{F}$, (I, S) is a YES-instance of SHCLA.

An algorithm to solve SHCLA

- Try all $j \in [r]$ so that $\mathcal{V}^{(j)}$ is the big part: assume $\partial\mathcal{V}^{(j)} \subseteq S \subseteq \mathcal{V}^{(j)}$.

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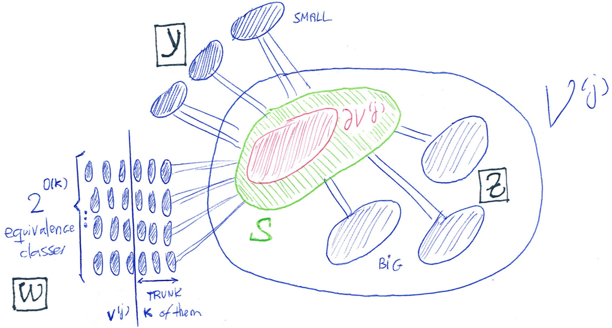
- Partition the connected components of $G \setminus S$ into 3 sets:
  - $\mathcal{Y}$: those that cannot go entirely in $\mathcal{V}^{(j)}$.
  - $\mathcal{Z}$: those that are big $(> k \cdot f_1(k))$ and that can go entirely in $\mathcal{V}^{(j)}$.
  - $\mathcal{W}$: those that are small $(\leqslant k \cdot f_1(k))$ and that can go entirely in $\mathcal{V}^{(j)}$.



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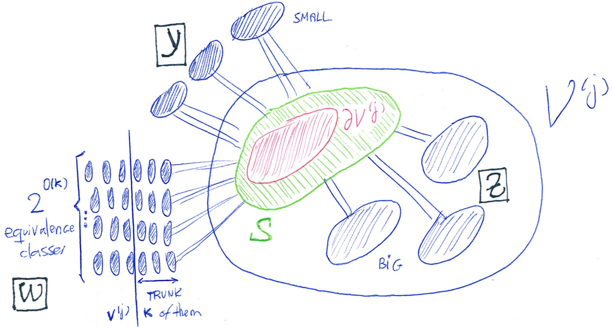


Lemma

The SHCLA problem can be solved in time $2^{O(k^2 \cdot \log k)} \cdot n$.

Piecing everything together

List Allocation (LA)

↓ FPT reduction

Connected List Allocation (CLA)

↓ FPT reduction

Highly Connected List Allocation (HCLA)

↓ FPT reduction

Split Highly Connected List Allocation (SHCLA)

↓ FPT algorithm to solve SHCLA

Theorem

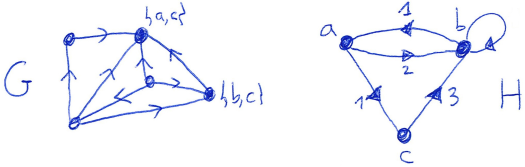
List Allocation *can be solved in time* $2^{O(k^2 \log k)} \cdot n^4 \cdot \log n$.

ARC-BOUNDED LIST DIGRAPH HOMOMORPHISM

Input: Two digraphs G and H, a list $\lambda : V(G) \rightarrow 2^{V(H)}$ of allowed images for every vertex in G, and a function α prescribing the number of arcs in G mapped to each (non-loop) arc of H.

Parameter: $k = \sum \alpha$.

Question: Decide whether there exists a homomorphism from G to H respecting the constraints imposed by λ and α.



- It generalizes several homomorphism problems. [Díaz, Serna, Thilikos '08]

Parameterization of Digraph Homomorphism

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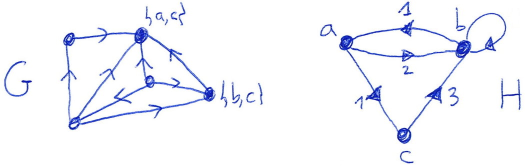

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Corollary

The Arc-Bounded List Digraph Homomorphism *problem is* FPT.

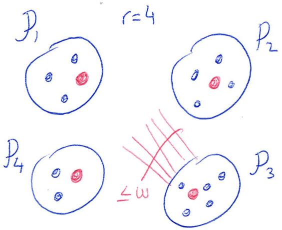
Graph partitioning problem

MIN-MAX GRAPH PARTITIONING

Input: An undirected graph G, $w, r \in \mathbb{Z}_{\geqslant 0}$, and $T \subseteq V(G)$ with $|T| = r$.

Parameter: $k = w \cdot r$.

Question: Decide whether there exists a partition $\{\mathcal{P}_1, \ldots, \mathcal{P}_r\}$ of $V(G)$ s.t. $\max_{i \in [r]} |\delta(\mathcal{P}_i, V(G) \setminus \mathcal{P}_i)| \leqslant w$ and for every $i \in [r]$, $|\mathcal{P}_i \cap T| = 1$.



- Important in approximation. [Bansal, Feige, Krauthgamer, Makarychev, Nagarajan, Naor, Schwartz'11]
- The "MIN-SUM" version is exactly the MULTIWAY CUT problem. [Marx '06]

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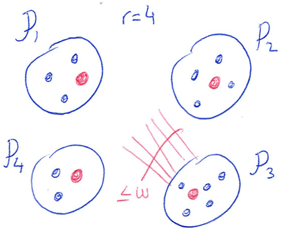

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- The "Min-Sum" version is exactly the Multiway Cut problem. [Marx '06]

Corollary

The Min-Max Graph Partitioning *problem is* FPT.

2-approximation for TREE-CUT WIDTH

- Tree-cut width is a graph invariant fundamental in the structure of graphs not admitting a fixed graph as an immersion. [Wollan '14]
- Tree-cut decompositions are a variation of tree decompositions based on edge cuts instead of vertex cuts.
- Tree-cut width also has algorithmic applications. [Ganian, Kim, Szeider '14]

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Corollary

There exists an algorithm that, given a graph G and a $k \in \mathbb{Z}_{\geqslant 0}$, in time $2^{O(k^2 \cdot \log k)} \cdot n^5 \cdot \log n$ either outputs a tree-cut decomposition of G with width at most $2k$, or correctly reports that the tree-cut width of G is strictly larger than k.

Conclusions and further research

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LIST ALLOCATION *can be solved in time* $2^{O(k^2 \log k)} \cdot n^4 \cdot \log n$.

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Some further research:

- Improve the running time of our algorithms.
- Can we find more applications of LIST ALLOCATION?
- Find an explicit (exact) FPT algorithm for tree-cut width.

Gràcies!