Hitting minors on bounded treewidth graphs

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[arXiv 1704.07284]

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Monadic Second Order Logic (MSOL):
Graph logic that allows quantification over sets of vertices and edges.

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Theorem (Courcelle, 1990)

Every problem expressible in MSOL can be solved in time $f(\text{tw}) \cdot n$ on graphs on n vertices and treewidth at most tw.

In parameterized complexity: FPT parameterized by treewidth.

Examples: Vertex Cover, Dominating Set, Hamiltonian Cycle, Clique, Independent Set, k-Coloring for fixed k, ...

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Major goal find the smallest possible function $f(\text{tw})$.

This is a very active area in parameterized complexity.

Remark: Algorithms parameterized by treewidth appear very often as a "black box" in all kinds of parameterized algorithms.

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But for the so-called connectivity problems, like Longest Path or Steiner Tree, the "natural" DP algorithms provide only time

$$2^{\mathcal{O}(\mathrm{tw} \cdot \log \mathrm{tw})} \cdot n^{\mathcal{O}(1)}.$$

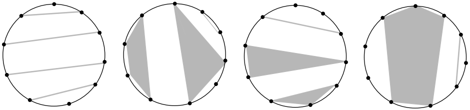
(Single-exponential algorithms on sparse graphs)

On topologically structured graphs (planar, surfaces, minor-free), it is possible to solve connectivity problems in time $2^{\mathcal{O}(\text{tw})} \cdot n^{\mathcal{O}(1)}$:

- Planar graphs: [Dorn, Penninkx, Bodlaender, Fomin. 2005]
- Graphs on surfaces: [Dorn, Fomin, Thilikos. 2006]
  [Rué, S., Thilikos. 2010]
- Minor-free graphs: [Dorn, Fomin, Thilikos. 2008]
  [Rué, S., Thilikos. 2012]

Main idea special type of decomposition with nice topological properties:

partial solutions $\Longleftrightarrow$ non-crossing partitions



$$\text{CN}(k) = \frac{1}{k+1}\binom{2k}{k} \sim \frac{4^k}{\sqrt{\pi}k^{3/2}} \leq 4^k.$$

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Deterministic algorithms with algebraic tricks: [Bodlaender, Cygan, Kratsch, Nederlof. 2013]

Representative sets in matroids: [Fomin, Lokshtanov, Saurabh. 2014]

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Cycle Packing: find the maximum number of vertex-disjoint cycles.

An algorithm in time $2^{\mathcal{O}(\text{tw} \cdot \log \text{tw})} \cdot n^{\mathcal{O}(1)}$ is optimal under the ETH.

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ETH: The 3-SAT problem on n variables cannot be solved in time $2^{o(n)}$

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There are other examples of such problems...

The $\mathcal{F}$-M-DELETION problem

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Question: Does G contain a set $S \subseteq V(G)$ with $|S| \leq k$ such that $G - S$ does not contain any of the graphs in $\mathcal{F}$ as a minor?

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  Solvable in time $2^{\Theta(\text{tw} \cdot \log \text{tw})} \cdot n^{\mathcal{O}(1)}$. [Jansen, Lokshtanov, Saurabh. 2014 + Pilipczuk. 2017]

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FPT by Courcelle, or by Graph Minors theory.

Goal of this project

Objective

Determine, for every fixed $\mathcal{F}$, the (asymptotically) smallest function $f_{\mathcal{F}}$ such that $\mathcal{F}$-M-DELETION/$\mathcal{F}$-TM-DELETION can be solved in time

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on n-vertex graphs.

- We do not want to optimize the degree of the polynomial factor.
- We do not want to optimize the constants.
- Our hardness results hold under the ETH.

Summary of our results

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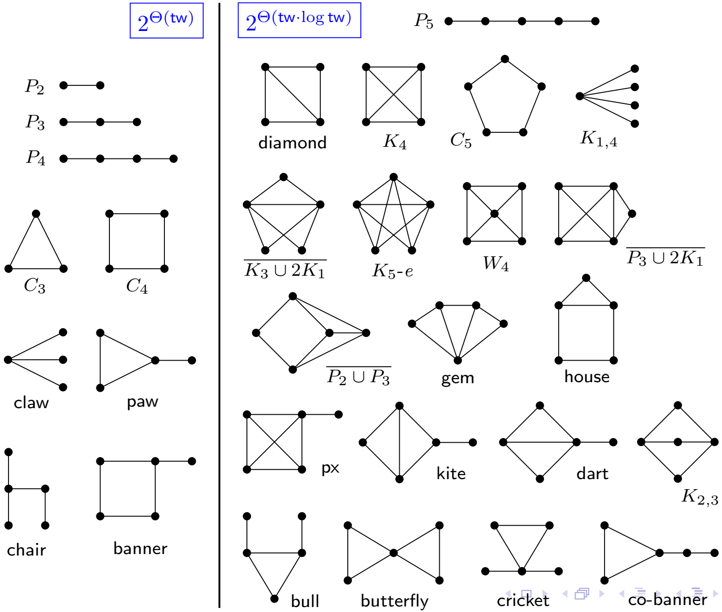
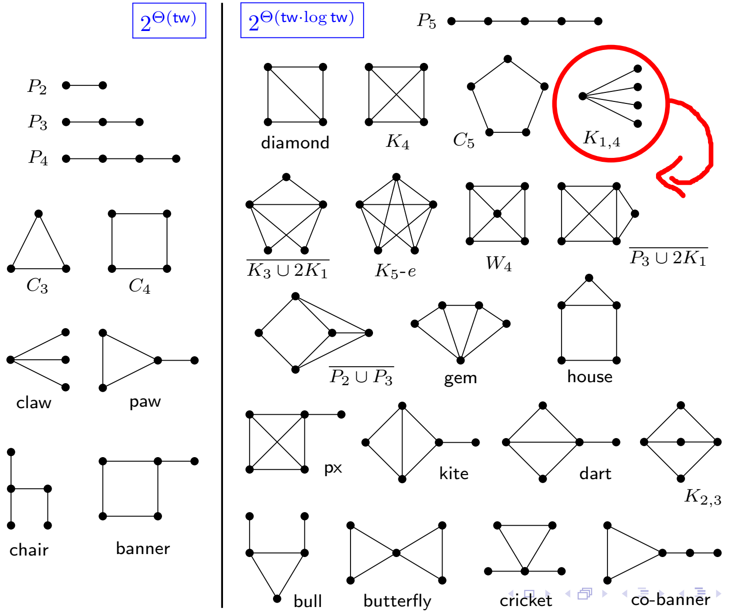
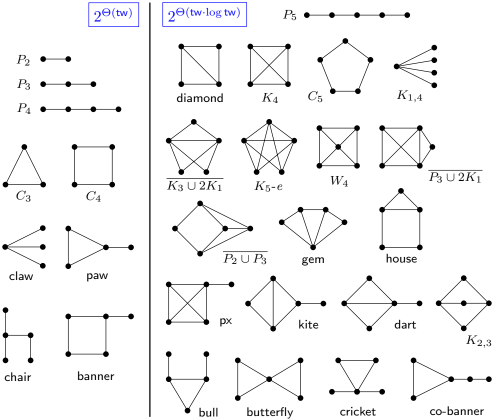
Complexity of hitting small planar minors H



For topological minors, there (at least) one change



A compact statement for small planar minors



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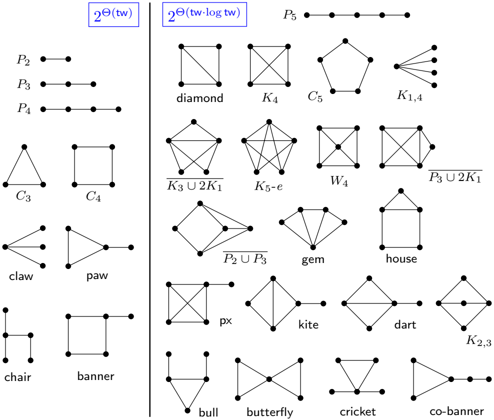

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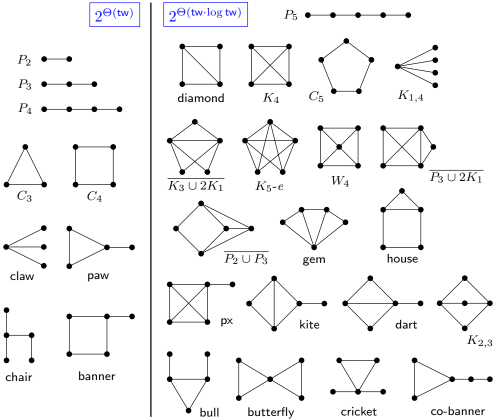

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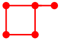
- All the graphs on the left are minors of  (called the banner)
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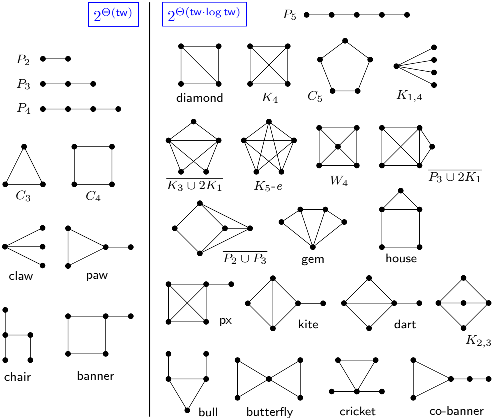

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- All the graphs on the right are not minors of  ... except P_5.

A dichotomy for hitting connected planar minors

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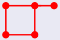
- $2^{\mathcal{O}(\mathrm{tw})} \cdot n^{\mathcal{O}(1)}$, *if* $H \preceq_m$  *and* $H \neq P_5$.

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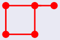
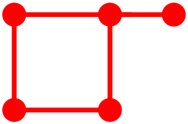
- $2^{\mathcal{O}(\mathrm{tw})} \cdot n^{\mathcal{O}(1)}$, *if* $H \preceq_{\mathrm{m}}$  *and* $H \neq P_5$.
- $2^{\mathcal{O}(\mathrm{tw} \cdot \log \mathrm{tw})} \cdot n^{\mathcal{O}(1)}$, *otherwise.*

In both cases, the running time is asymptotically optimal under the ETH.

Why the banner??



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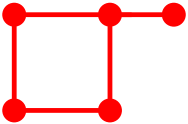

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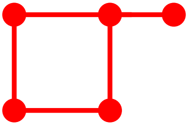

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  - a cycle (of any length),
  - or a tree in which some vertices have been replaced by triangles.

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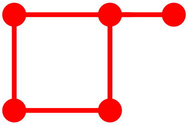

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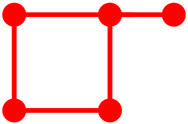
Why the banner??



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- If the characterization of the allowed connected components is enriched in some way, such as restricting the length of the allowed cycles or forbidding certain degrees, the problem becomes harder.

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3. Lower bounds under the ETH
   - $2^{o(\text{tw})}$ is "easy".
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     [Lokshtanov, Marx, Saurabh. 2011] [Marcin Pilipczuk. 2017] [Bonnet, Brettell, Kwon, Marx. 2017]

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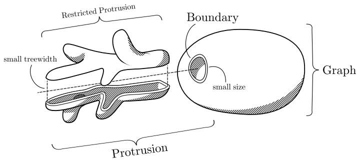
We build on the machinery of boundaried graphs and representatives:



[Bodlaender, Fomin, Lokshtanov, Penninkx, Saurabh, Thilikos. 2009]

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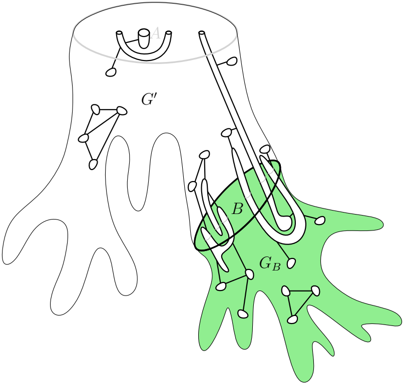

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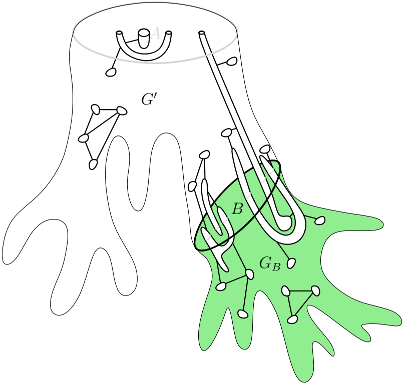

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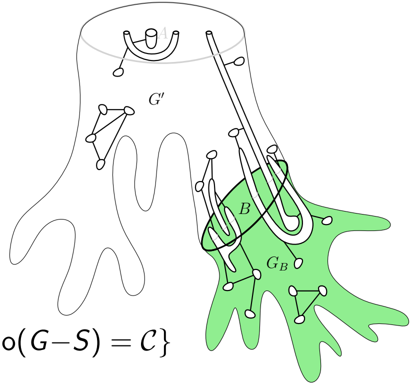

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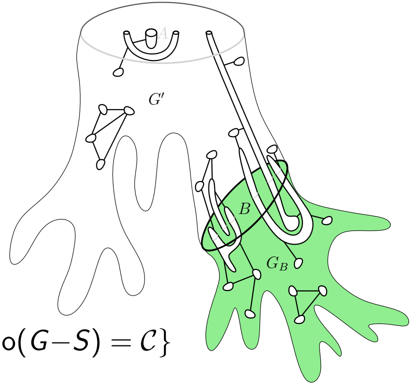

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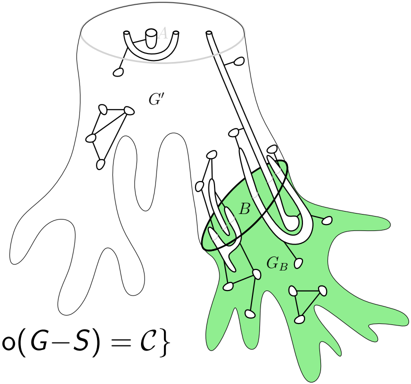

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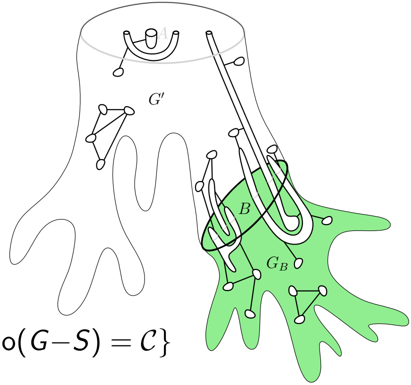

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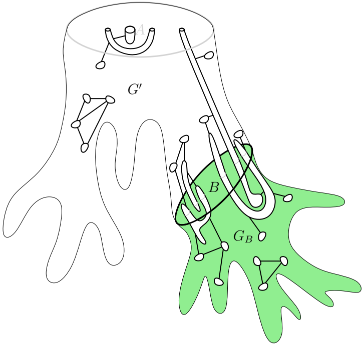

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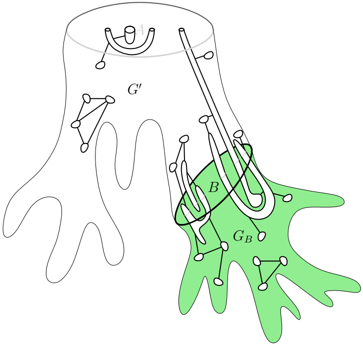

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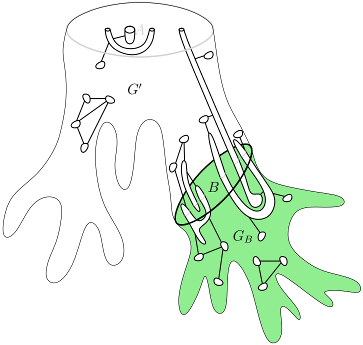

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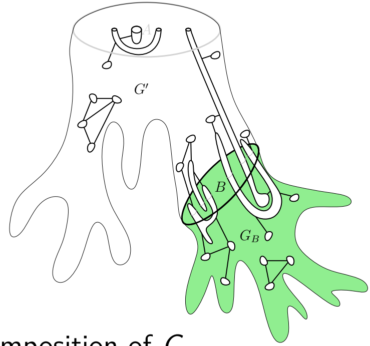

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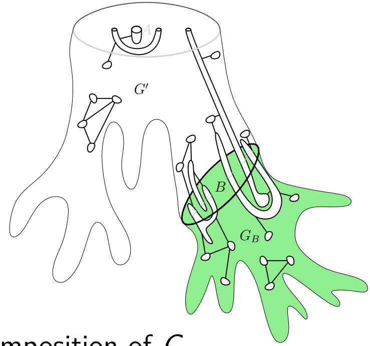

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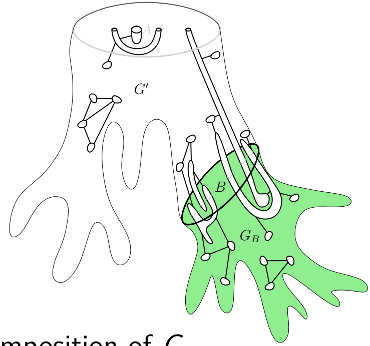

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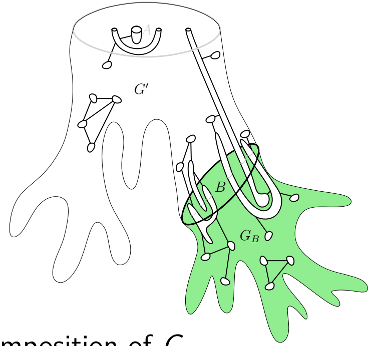

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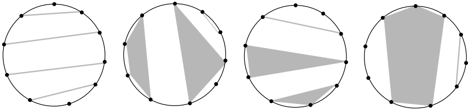

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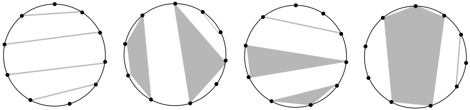

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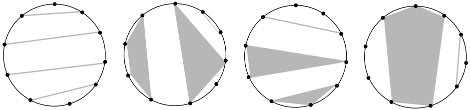

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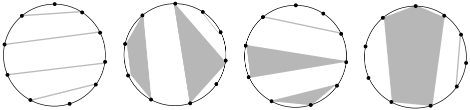

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- We can extend this algorithm to input graphs G embedded in arbitrary surfaces by using surface-cut decompositions. [Rué, S., Thilikos. 2014]

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- Consider families $\mathcal{F}$ containing disconnected graphs.

Gràcies!