The number of labeled graphs of bounded treewidth

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Séminaire COATI
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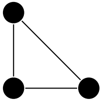
[arXiv 1604.07273]

Next section is...

Example of a 2-tree:



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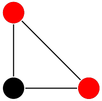

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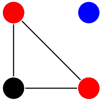

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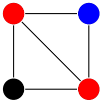

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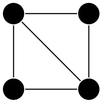

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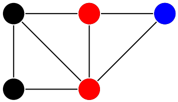

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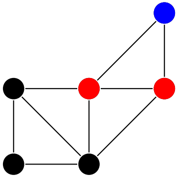

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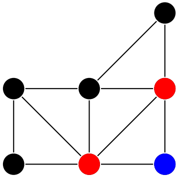

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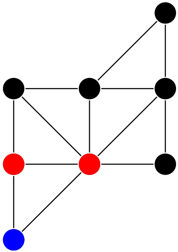

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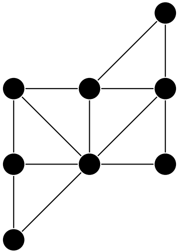

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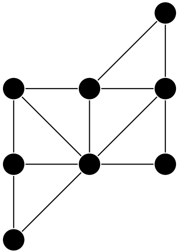

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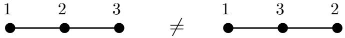
A partial k-tree is a subgraph of a k-tree.

A graph has treewidth at most k if and only if it is a partial k-tree.

What is known about the number of (partial) k-trees?

Labeled k-trees



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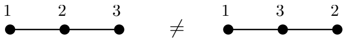

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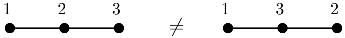

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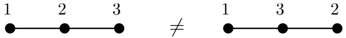

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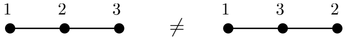

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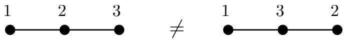

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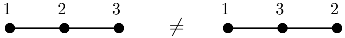

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- Nothing was known for general k.

$T_{n,k}$ and an easy upper bound

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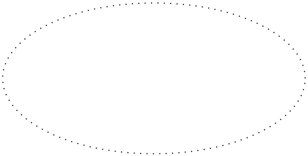
An easy lower bound



Take a forest on $n - (k-1)$ vertices:
$(n-k+1)^{(n-k-1)}$ possibilities

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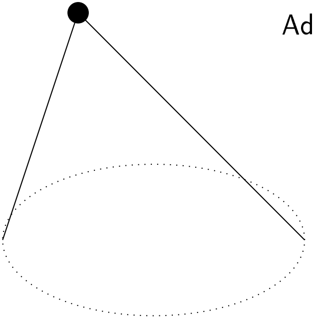

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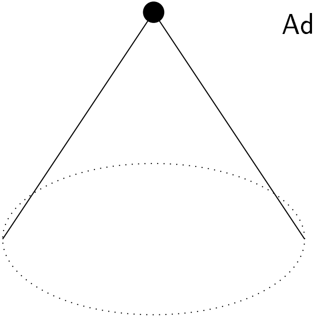

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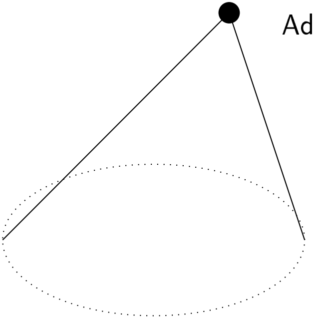

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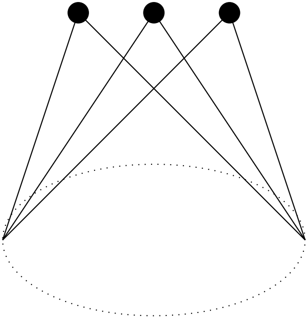

Add $k-1$ vertices connected to the forest: $2^{(k-1)(n-(k-1))}$ possibilities

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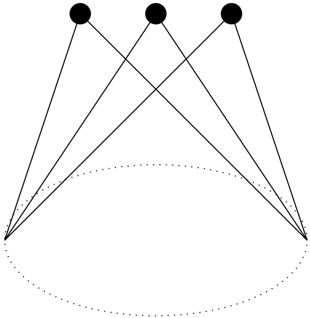

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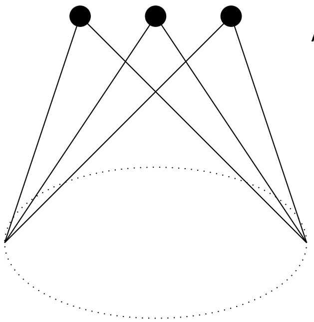

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$$T_{n,k} \geq (n-k+1)^{(n-k-1)} \cdot 2^{(k-1)(n-k+1)} \geq \left(\frac{1}{4} \cdot 2^k \cdot n\right)^n \cdot 2^{-k^2}$$

Our results

Summarizing, so far we have:

$$T_{n,k} \leq (k \cdot 2^k \cdot n)^n \cdot 2^{-\frac{k(k+1)}{2}} \cdot k^{-k}$$

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For any two integers n, k with $1 < k \leq n$, the number $T_{n,k}$ of n-vertex labeled graphs with treewidth at most k satisfies

$$T_{n,k} \geq \left(\frac{1}{128e} \cdot \frac{k \cdot 2^k \cdot n}{\log k}\right)^n \cdot 2^{-\frac{k(k+3)}{2}} \cdot k^{-2k-2}.$$

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Trade-off creating many graphs vs. bounding the number of duplicates

Some ingredients of the construction:

1. labeling function σ: permutation of $\{1, \ldots, n\}$ with $\sigma(1) = 1$.
2. We will introduce vertices $\{v_1, v_2, \ldots, v_n\}$ one by one following the order $v_{\sigma(1)}, v_{\sigma(2)}, \ldots, v_{\sigma(n)}$.
3. If $j < i$, the vertex $v_{\sigma(j)}$ is said to be to the left of $v_{\sigma(i)}$.

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Proper linear k-trees: graphs that can be constructed starting from a $(k+1)$-clique and iteratively adding a vertex v_i connected to a clique K_{v_i} of size k (called the active vertices), with the constraints that

- $v_{i-1} \in K_{v_i}$.
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The graphs G we will construct satisfy $\mathbf{tw}(G) \leq \mathbf{pw}(G) \leq \mathbf{ppw}(G) \leq k$.

Ingredients of the construction

For every $i \geq k+1$ we define:

1. A set $A_i \subseteq \{j \mid j < i\}$ with $|A_i| = k$ of active vertices (as in the definition of proper linear k-trees).

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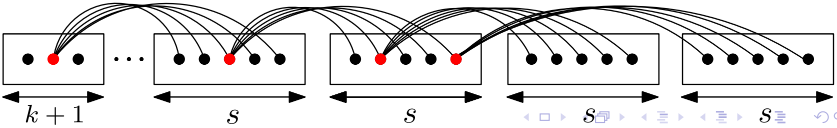
4. We insert the vertices by consecutive blocks of size $s = s(n,k)$. We will fix the value of s later.
5. A vertex $a_i \in A_i$, called the anchor: all vertices of the same block are adjacent to the same anchor a_i.

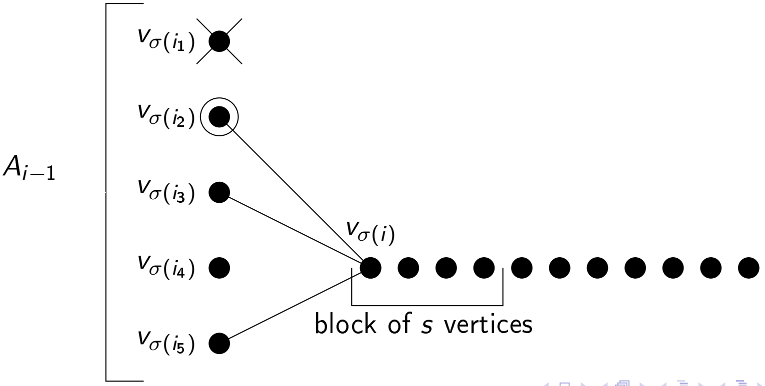


Description of the construction

1. Choose σ, a permutation of $\{1, \ldots, n\}$ such that $\sigma(1) = 1$.
2. Choose the first $(k+1)$-clique, with $1 \in N(i)$ for $2 \leq i \leq k+1$.
3. Define $a_{k+1} = 1$.

Description of the construction

1. If $i \equiv k+2 \pmod{s}$ (that is, at the beginning of a block):
   - Define $f(i) = a_{i-1}$.
   - Define $A_i = (A_{i-1} \setminus \{f(i)\}) \cup \{i-1\}$.
   - Define $a_i = \min A_i$.
   - Choose $N(i) \subseteq A_i$ such that $a_i \in N(i)$ and $|N(i)| > \frac{k+1}{2}$.



Description of the construction

1. If $i \not\equiv k+2 \pmod{s}$ (that is, at the middle of a block):
   - Choose $f(i) \in (A_{i-1} \setminus \{a_{i-1}\}) \cap N(i-1)$.
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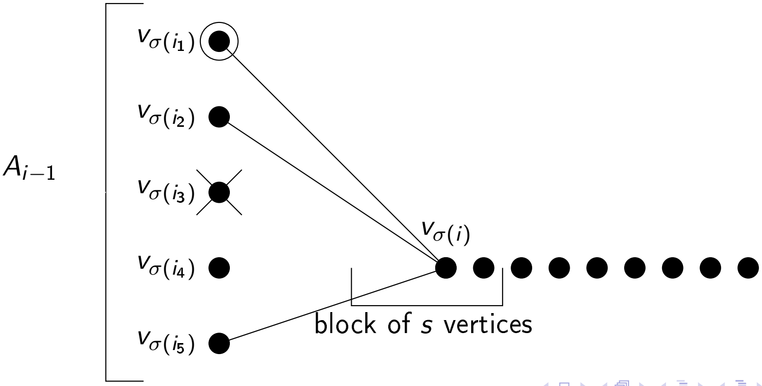

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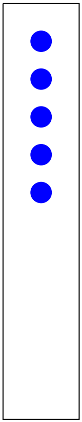

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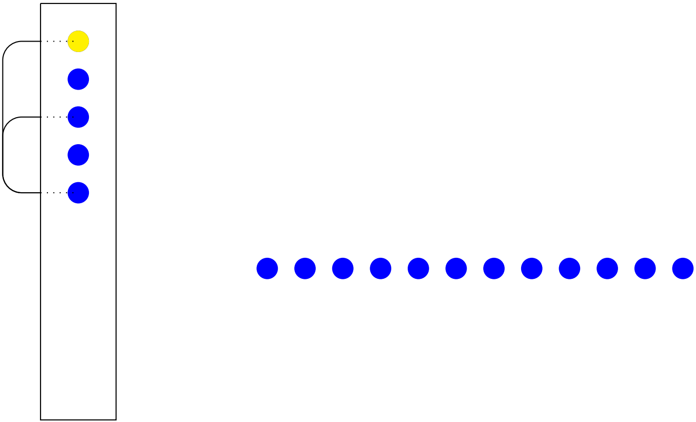

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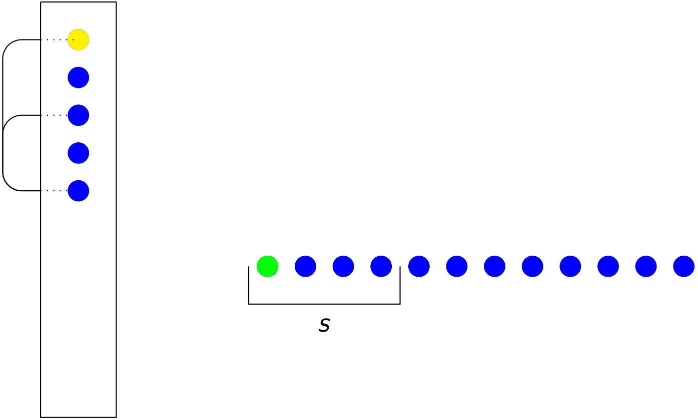

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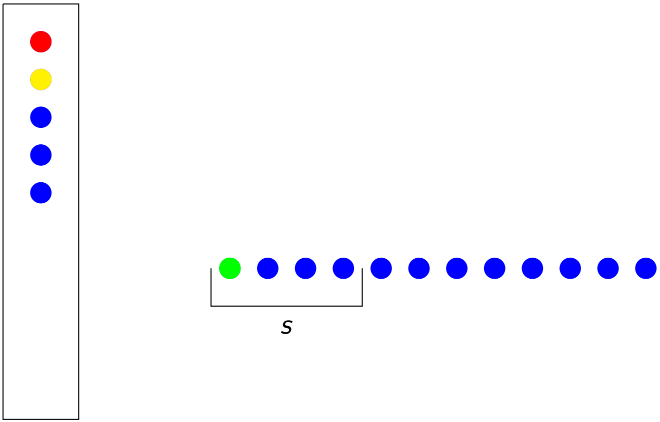

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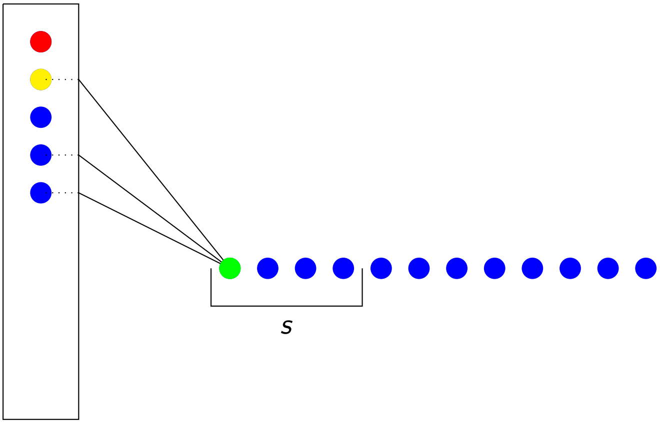

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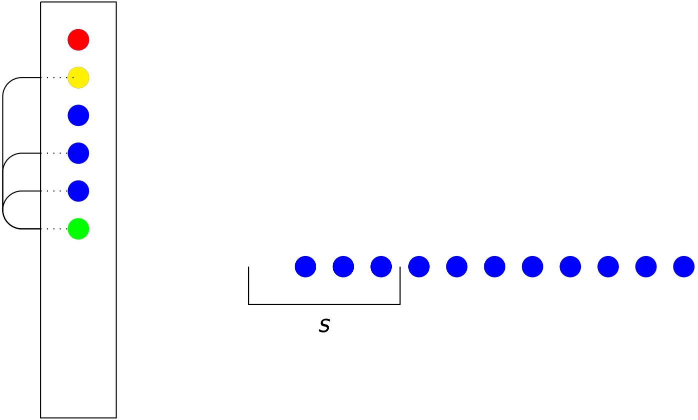

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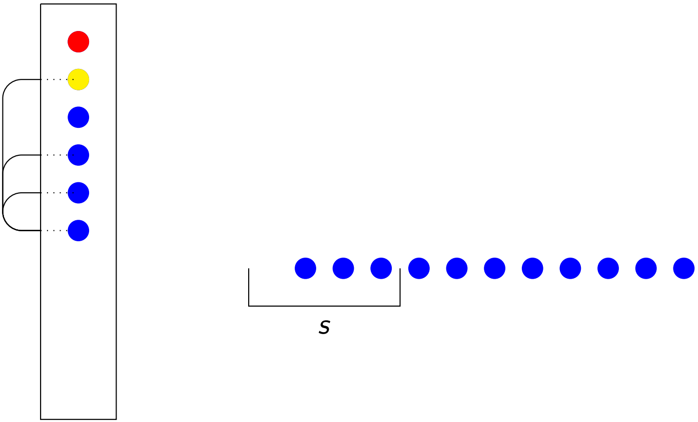

Example of the construction (by Julien Baste)

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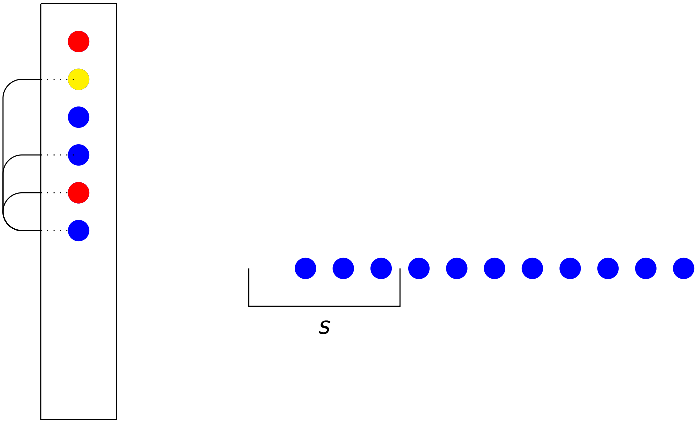

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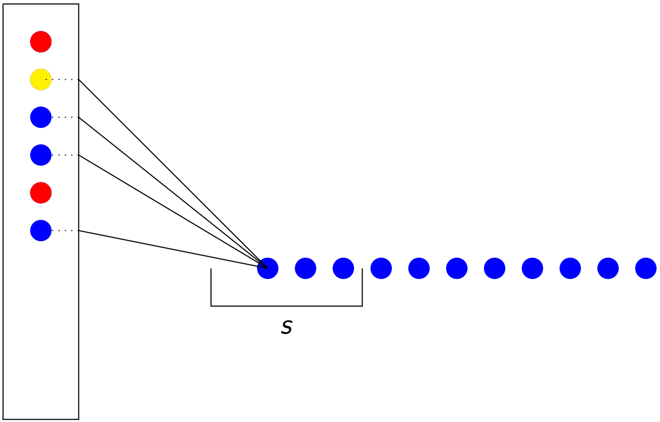

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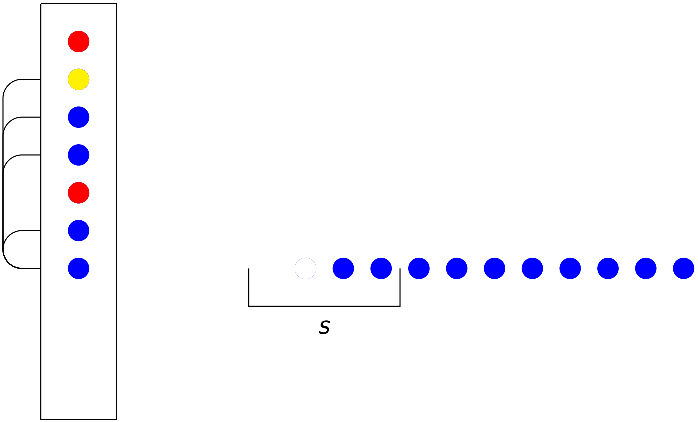

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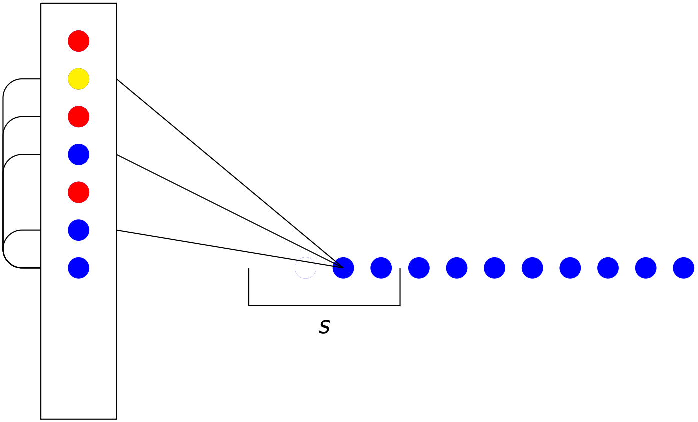

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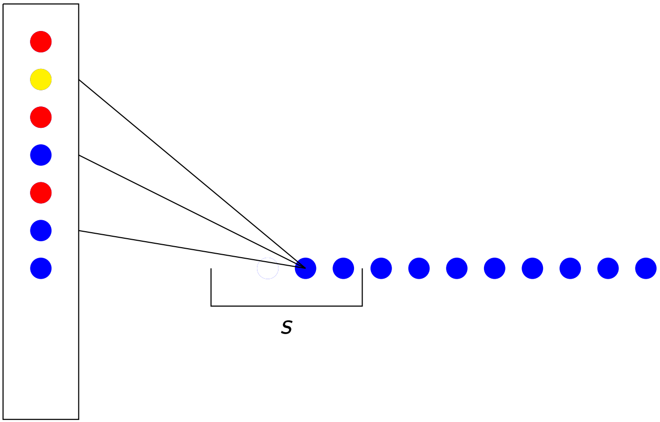

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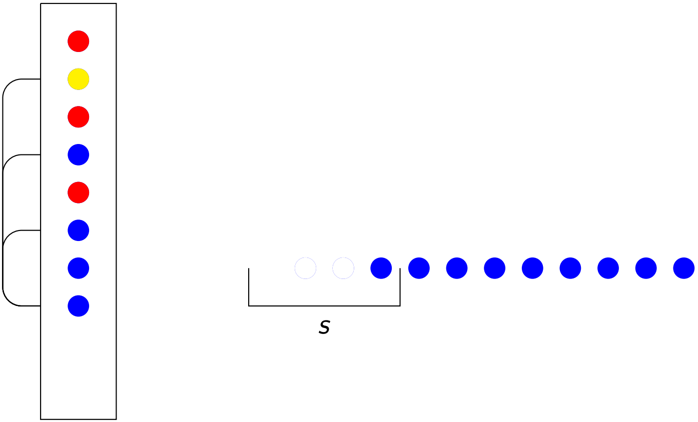

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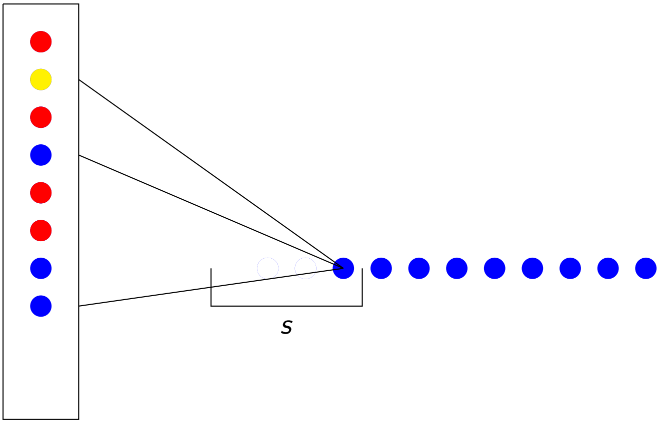

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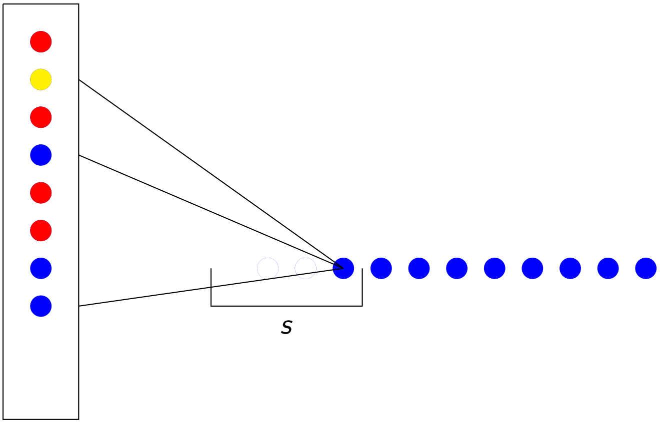

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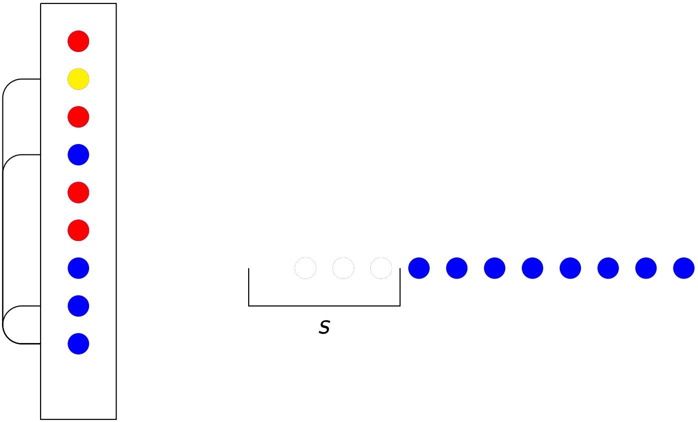

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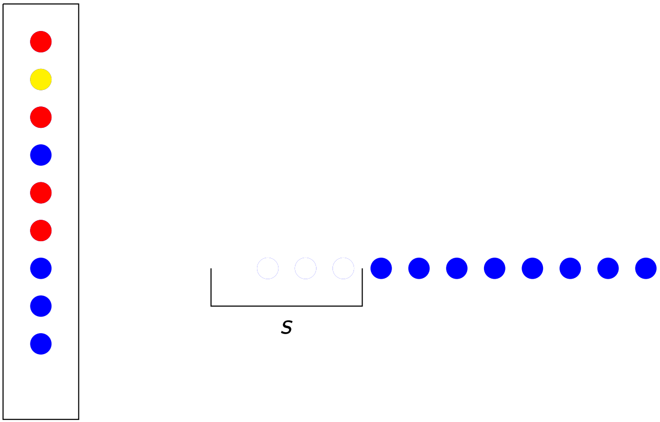

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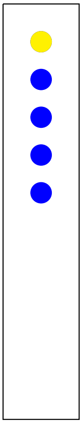

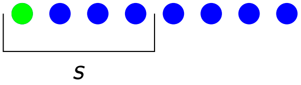

Next section is...

Analysis of the construction

First note that the graphs G we construct indeed satisfy $\mathbf{ppw}(G) \leq k$.

Analysis of the construction

- How many graphs are created by the construction?
- How many times the same graph may have been created?

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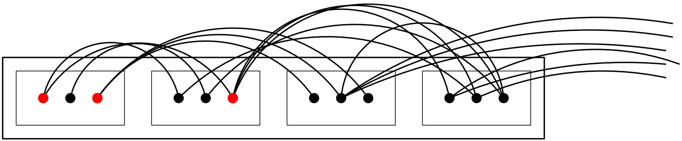

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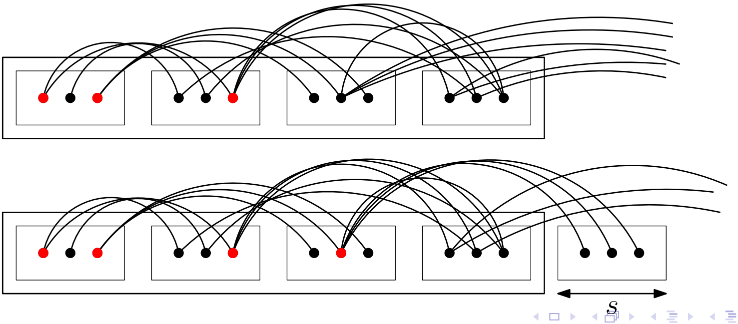

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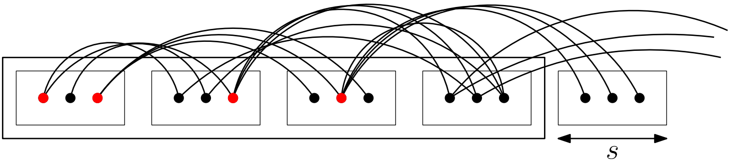

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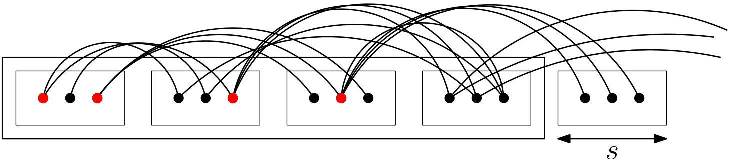

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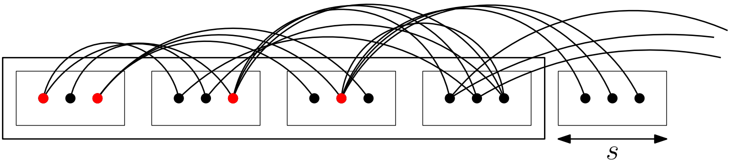

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So $s(n, k) = \log k$ is the best choice for the block size.

Next section is...

Further research

$$T_{n,k} \leq (k \cdot 2^k \cdot n)^n \cdot 2^{-\frac{k(k+1)}{2}} \cdot k^{-k}.$$

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- Improve the upper bound for pathwidth or proper-pathwidth?
- Other relevant parameters: branchwidth, cliquewidth, rankwidth, tree-cutwidth, booleanwidth, ...

Gràcies!